Chapter 2

Schemes: First Steps

In this chapter, we will look at the basic object of study in modern algebraic geometry: schemes.

Section 2.1 The Spectrum of a Ring

Definition 2.1.1. Let R be a commutative ring (as always, with 1). As a set, we define the *spectrum* of R, written $Spec(R)$, to be the collection of prime ideals $P \subset R$.

Example 2.1.2.

(a) $Spec(k)$ consists of a single point if k is a field.
(b) $Spec(\mathbf{Z}) = \{0, 2, 3, 5, 7, \ldots\}$.
(c) If k is an algebraically closed field, then $Spec(k[t]) = \{*\} \cup k$.

Definition 2.1.3. Let $f \in R$ be an arbitrary element. Let $P \in X = Spec(R)$ be a point. Identify P with a prime ideal in R. Define $f(P)$ to be the residue class of f in the quotient ring R/P.

Remark 2.1.4. The previous definition allows us to think of an element $f \in R$ as a "function" on $X = Spec(R)$, in the sense that we have "values" $f(P)$ defined for each point $P \in X$. However, these functions have the somewhat disturbing property that the space where their values live varies depending on the point where we're doing the evaluating. The only reason we weren't bothered earlier—in the case where R is a finitely generated algebra without nilpotents over an algebraically closed field k—is that the value of a regular function $f \in A(X)$ at a point x corresponding to a maximal ideals M lives in the quotient ring $A(X)/M$, and all of these quotients can be canonically identified with the field k.

Lemma 2.1.5: *Let R be a commutative ring, $f \in R$ an element, and $P \subset R$ a prime ideal. Then $f(P) = 0$ if and only if $f \in P$.*

Proof: Obvious. ∎

Definition 2.1.6. Let $X = Spec(R)$. Let $S \subset R$ be an arbitrary subset. We define the *zero set* of S to be

$$Z(S) = \{P \in X : f(P) = 0 \forall f \in S\} = \{P \in X : P \supset S\}$$

Remark 2.1.7. As before, one can check that the collection consisting of all zero sets of functions satisfies the requisite properties to be the closed sets in a topology on X; the resulting topology on the spectrum of a ring is called the *Zariski topology* .

Definition 2.1.8. Let $X = Spec(R)$. Let $Y \subset X$ be an arbitrary subset. Define the ideal of functions that vanish on Y to be

$$I(Y) = \{f \in R : f(P) = 0 \forall P \in Y\} = \{f \in R : f \in P \forall P \in Y\} = \cap_{P \in Y} P.$$

Lemma 2.1.9: *For any ideal J in a commutative ring R, the radical of J is*

$$\{f \in R : \exists N > 0, f^N \in J\} = \cap_{P \supset J} P.$$

Proof: We have defined the radical to be the set on the left-hand side of the equality. It is easy to see that the radical is contained in the set on the right-hand side. Conversely, suppose that $f \in R$ is not contained in the radical. Consider the collection T of all ideals of R that contain J and are disjoint from the multiplicative set $S = \{f^N, N \geq 1\}$. By hypothesis, J lies in the collection T. By Zorn's Lemma, T contains a maximal element Q. We claim that Q is a prime ideal. To see this, suppose f_i (for $i = 1, 2$) lives in R but not in Q, and that $f_1 f_2 \in Q$. By maximality, $Q+ < f_i >$ meets S; let's say it contains an element $q_i + e_i f_i$ with $q_i \in Q$. Since S is a multiplicative set, it contains the product

$$(q_1 + e_1 f_1)(q_2 + e_2 f_2).$$

However, under the hypotheses, this product is also contained in Q. This contradicts the fact that Q does not meet S, and thus shows that Q is a prime ideal. Therefore, no power of f can be contained on the right-hand side, proving the result. ∎

Lemma 2.1.10: *Let $X = Spec(R)$*

(i) If J is a proper ideal in R, then $Z(J)$ is nonempty.
(ii) For any ideal $J \subset R$, one has $I(Z(J))$ equal to the radical of J.
(iii) For any $Y \subset X$, one has $Z(I(Y))$ equal to the closure of Y in the Zariski topology.

Proof: Item (i) follows from Zorn's Lemma; item (ii), from the previous lemma; item (iii), directly from the definitions. ∎

Remark 2.1.11. The first item is the analog of the weak Nullstellensatz; the second item is the analog of Hilbert's Nullstellensatz.

Remark 2.1.12. The Zariski topology is almost never Hausdorff. For example, suppose R is an integral domain. Then the zero ideal $< 0 >$ is prime in R. So, it can be identified with a point $\eta \in X$, usually called the *generic point* of X. Observe first that $I(\eta) = 0$, this being the intersection of the prime ideals that contain 0. Next, the associated zero set $Z(0)$ is the whole space, since every prime ideal contains the zero ideal. So, the point η is not only non-closed, but actually dense in the entire spectrum of R. In general, the only thing you can say is that the spectrum of a ring is T_0; i.e.., given a pair of points, there exists a neighborhood of one not containing the other.

Definition 2.1.13. Let R be a commutative ring, and let $f \in R$. Define the *standard open set* corresponding to f to be the complement of its zero set; that is, $D(f) = Spec(R) \backslash Z(f)$.

Proposition 2.1.14: *The standard open sets in $Spec(R)$ form a basis for the topology.*

Proof: This is the same as the proof given previously that the affine open algebraic sets form a basis for the topology on an affine variety. ∎

Definition 2.1.15. A topological space X is called *quasicompact* if every open cover of X contains a finite subcover.

Remark 2.1.16. Quasicompactness isn't terribly useful without the Hausdorff axiom to accompany it. Later, we'll consider categorical notions (separated, proper) that provide good substitutes for some standard topological notions (Hausdorff, compact).

Proposition 2.1.17: *Let $X = Spec(R)$. Then X is quasicompact.*

Proof: Given any open cover of X, we can refine it to a cover in which every set is a standard open sets $D(f_i)$, for $i \in I$. Since the union of the $D(f_i)$ covers X, the intersection of the complements

$Z(f_i)$ is empty. Thus, $Z(< f_i, i \in I >) = \emptyset$. We've seen that the zero zet of a proper ideal can't be empty, so the condition that a collection of standard open sets covers X reduces to

$$R =< f_i, i \in I >$$

or to the existence of a representation

$$1 = \sum e_i f_i$$

for some $e_i \in R$. But any such representation must actually be a *finite* sum; we only need to retain the open sets in the cover for which f_i actually appears with a nonzero coefficient in this representation. ∎

In a little while, we 're going to need a strengthened version of this result: the standard open sets $D(f)$ are themselves quasicompact. Let's see how to spice up the current result to get the one we really need.

Lemma 2.1.18: *Let $f, g \in R$, and let N be a nonnegative integer. Then*

(i) $D(f) \cap D(g) = D(fg)$; *and*
(ii) $D(f^N) = D(f)$.

Proof: $D(f)$ is the set of points where f is nonzero. In other words, it is the set of prime ideals not containing f. Now the result follows by translating the definition of prime ideal into the current context. ∎

Proposition 2.1.19: *Every standard open set $D(f)$ in $X = Spec(R)$ is quasicompact.*

Proof: Take an open cover. By refining the cover, we can assume all the open sets in the cover are themselves standard open sets, of the form $D(f_i)$ for i ranging through some index set I. Taking the complement of the covering relation, we find that

$$Z(f) \supset \cap_i Z(f_i) = Z(< f_i, i \in I >).$$

Now we can reverse the containment by passing back to the ideals of functions that vanish on these closed sets, obtaining

$$f \in I(Z(f)) \subset I(Z(< f_i, i \in I >)) = Rad(< f_i, i \in I >).$$

So, the covering condition reduces to the existence of a nonnegative integer N and a relation

$$f^N = \sum e_i f_i$$

for some $e_i \in R$. Once again, any such relation must be *finite* ; we just keep the sets in the open cover where the coefficients are nonzero. ∎

Remark 2.1.20. The kinds of relations that we've seen twice in the proof of quasicompactness are the algebraic analogs of partitions of unity. They can (and will) be used in similar ways.

Section 2.2 Affine Schemes

We're on the verge of being able to define an affine scheme. It's more than the set $Spec(R)$; it's even more than the Zariski topology on this set; it's all that plus a structure sheaf.

Definition 2.2.1. Let $X = Spec(R)$ be the spectrum of a commutative ring. We define the *structure sheaf* on X to be the sheaf $\mathcal{O}_X$ whose ring of sections $\mathcal{O}_X(U)$ on an open $U \subset X$ consists of those functions

$$s : U \to \coprod_{P \in U} R_P$$

that satisfy the following properties:

(i) for all $P \in U$, one has $s(P) \in R_P$; and

(ii) for all $P \in U$, there exists an open neighborhood $P \in V \subset U$ and elements $f, g \in R$ such that for all $Q \in V$, one has $f \notin Q$ and $s(Q) = g/f \in R_Q$.

Remark 2.2.2. This definition tells us the sections of the structure sheaf over open sets. It leaves us to infer how to define the restriction maps to get a presheaf structure. (Since the sections are actual functions of actual sets, you can just restrict them in the usual way.) It also leaves us to infer that the resulting presheaf actually satisfies the sheaf axioms. This is reasonably straightforward to check: In terms of its logical structure (in particular, the number and placement of quantifiers), the definition of the structure sheaf precisely parallels the definition we used when constructing the sheaf associated to a presheaf. By analogy, one also suspects that the local rings R_P should become isomorphic to the stalks of the structure sheaf. The only way to verify this, however, is to get a better understanding of the ring of sections over some interesting open sets.

Lemma 2.2.3: *Let $X = Spec(R)$ and let $f \in R$. There is a natural ring homomorphism $\alpha_f : R_f \to \mathcal{O}_X(D(f))$.*

Proof: An element of R_f can always be represented as a fraction g/f^N with $g \in R$ and N a nonnegative integer. If $Q \in D(f) = D(f^N)$, then $f^N \notin Q$, and so the fraction g/f^N determines a well-defined element in each of the local rings R_Q. We define $\alpha_f(g/f^N)$ to be the function that assigns to a prime ideal Q the element $g/f^N \in R_Q$. ∎

Lemma 2.2.4: *The maps α_f are injective.*

Proof: Suppose $s = g/f^N \in R_f$ and $\alpha_f(s) = 0$. That means, for every prime ideal $Q \in D(f)$, the fraction g/f^N represents zero when considered as an element of the local ring R_Q. So, there exist elements h_Q (depending on Q, of course), such that $h_Q g = 0 \in R$. From this relation, we can conclude that the fraction g/f^N actually represents 0 as soon as both f and h_Q have been inverted; i.e., in R_{fh_Q}, and hence as functions on $D(f) \cap D(h_Q) = D(fh_Q)$. Without loss of generality, we can replace h_Q by fh_Q. Now we have such a relation for every point $Q \in D(f)$. So, the collection of standard open sets of the form $D(h_Q)$ is an open cover of $D(f)$. By the proof of quasicompactness, this means that we have a (finite partition of unity) relation of the form

$$f^M = \sum e_Q h_Q.$$

Multiplying this relation through by g, we get

$$f^M g = \sum e_Q h_Q g = \sum 0 = 0.$$

But that equation tells us that g/f^N already represents the zero element when viewed in the ring R_f, which was exactly what we needed to show. ∎

Proposition 2.2.5: *The maps α_f are isomorphisms.*

Proof: Since we've already proved that they are injective, we just need to verify surjectivity. So, take a section $s \in \mathcal{O}_X(D(f))$. By definition, near each point of $D(f)$ there is an open neighborhood on which s can be represented as a quotient of elements of R. Using the facts that the standard open sets form a basis for the topology and that $D(f)$ is quasicompact, we reduce to considering the following situation. We are given a finite collection of elements $f_i, g_i \in R$ and nonnegative integers m_i such that:

(i) The sets $D(f_i)$ cover $D(f)$;

(ii) The fractions $g_i/f_i^{m_i} \in R_{f_i}$ define sections in $\mathcal{O}_X(D(f_i))$ that agree when restricted to the intersections $D(f_i) \cap D(f_j)$ for different i and j.

The first simplification is to observe that we can replace the various exponents m_i appearing in the denominators with a single nonnegative integer M. (Just multiply numerator and denominator by the right power of f_i, which we can do by finiteness.) Using the fact that $D(f) = D(f^M)$, we can then make things even simpler by assuming that $M = 1$. Now the restrictions in question live in $\mathcal{O}_X(D(f_i f_j))$. The elements being restricted all live in the image of $\alpha_{f_i f_j}$. Since that map is injective, the second condition reduces to the statement that

$$\frac{g_i}{f_i} = \frac{g_j}{f_j} \in R_{f_i f_j}.$$

Translating this into a condition in R, we learn that there exist nonnegative integers N (depending on i and j, but we can ignore that because of finiteness) such that

$$(f_i f_j)^N (g_i f_j - g_j f_i) = 0 \in R.$$

Rewriting this expression a tad, we get

$$[g_i f_i^N][f_j^{N+1}] - [g_j f_j^N][f_i^{N+1}] = 0.$$

Replacing our original pairs of data (f_i, g_i) by the new pairs $(f_i^{N+1}, g_i f_i^N)$, we can then assume that $N = 0$ as well. After these simplifications, we can turn our attention to the first condition. Because we have a cover, the proof of quasicompactness gives us a partition of unity relation of the form

$$f^T = \sum e_i f_i.$$

Define

$$g = \sum e_i g_i.$$

Multiply by f_j and compute:

$$g f_j = \sum e_i g_i f_j = \sum e_i f_i g_j = f^T g_j \in R.$$

In other words, there is an equality of fractions $g/f^T = g_j/f_j$ in all of the rings R_{f_j} showing that $\alpha_f(g/f^T)$ must hit our original section s. ∎

Before stating some of the important corollaries of this result, I need to introduce yet another piece of notation.

Definition 2.2.6. Let F be a sheaf of abelian groups on a topological space X. For each open $U \subset X$, define $\Gamma(U, F) = F(U)$.

Remark 2.2.7. It seems sort of silly to take a simple notation like $F(U)$ and expand it to the more complicated looking $\Gamma(U, F)$. Why do we need two names for the same thing? Well, the first notation is particularly useful when we are thinking about a fixed sheaf F, and want to emphasize the fact that it defines a functor of the variable U. The second notation is more useful when we are considering more than one sheaf at a time, and want to emphasize how the group of sections for a fixed U (most often taking $U = X$ itself) changes as the sheaf varies.

Corollary 2.2.8: $\Gamma(D(f), \mathcal{O}_X) = R_f$.

Proof: This is a simple restatement of the proposition. ∎

Corollary 2.2.9: $\Gamma(X, \mathcal{O}_X) = R$.

Proof: This is a special case of the previous corollary, since $X = D(1)$. ∎

Corollary 2.2.10: *Let $x \in X = Spec(R)$ correspond to the prime ideal $P \subset R$. Then the stalk $\mathcal{O}_{X,x}$ is isomorphic to the local ring R_P.*

Proof: Because the standard open sets form a basis for the topology, we can compute the direct limit that defines the stalk by looking at the sections of $\mathcal{O}_X$ over those $D(f)$ that contain P. However, $P \in D(f)$ if and only if $f \notin P$. So, the stalk is the limit of $\mathcal{O}_X(D(f)) = R_f = R[1/f]$, where we end up inverting precisely the elements of $R \backslash P$. ∎

Remark 2.2.11. Let R be an integral domain, with generic point η. The stalk of the structure sheaf at the generic point is just the field of fractions of R.

Now let's look at a pair of rings and a homomorphism $\varphi : R \to S$ between them. Since we're supposed to think of elements of the rings as functions on the corresponding spectra, we can at least hope that φ is related in some sensible way to a map on spectra that goes in the other direction ($f : Spec(S) \to Spec(R)$). After a moment's reflection, we can see what this map should be on the level of sets: Given a prime ideal P in S, its inverse image $\varphi^{-1}(P)$ is a prime ideal in R.

Lemma 2.2.12: *Given a homomorphism of rings, $\varphi : R \to S$, it induces a function $f : Spec(S) \to Spec(R)$ on spectra, given by $f : P \mapsto \varphi^{-1}(P)$. Moreover, f is a continuous map of topological spaces.*

Proof: Under the map f, the inverse image $f^{-1}(Z(J))$ of the closed set defined by an ideal $J \subset R$ consists of the set of prime ideals $P \subset S$ such that $f(P) \in Z(J)$. Equivalently, $\varphi^{-1}(P) \supset J$ or $P \supset \varphi(J)$. In other words, the inverse image of $Z(J)$ is just the closed set $Z(\varphi(J))$. ∎

So far, we've defined a map on spectra, coming from a homomorphism, that reflects the topological part of the structure. What can we expect on the sheaf-theoretic part? Well, let's write $X = Spec(R)$ and $Y = Spec(S)$, and keep the rest of our notation the same, so that $\varphi : R \to S$ is a ring homomorphism that defines a continuous function $f : Y \to X$. Take an open subset $U \subset X$ and a section $s \in \mathcal{O}_X(U)$. Write, for the moment, $V = f^{-1}(U)$. By composing the set-theoretic functions that underly the whole structure, we get a function

$$s \circ f : V \to \coprod_{P \in U} R_P.$$

Let's now think about a point $Q \in V$. This point represents a prime ideal of S such that $P := f(Q) = \varphi^{-1}(Q)$. The ring homomorphism $\varphi : R \to S$ can be composed with the localization map $S \to S_Q$ to give a natural homomorphism $R \to S_Q$ which just happens to take every element of $R \backslash P$ to an invertible element of S_Q. By the universal property of localizations, we get a naturally induced homomorphism $R_P \to S_Q$. Composing yet again, we get

$$\text{``}\varphi\text{''} \circ s \circ f : V \to \coprod_{Q \in V} S_Q.$$

It is (relatively) easy to check that this composite represents a section of $\mathcal{O}_Y(f^{-1}(U))$.

There appear to be some technical difficulties with using this construction to understand how the structure sheaves on the two spectra are related. First, we haven't explained what to do with points $P \in U$ that are not in the image of f. Second, we've only described what's going on with sections on open sets of Y of the form $V = f^{-1}(U)$, and not on general open sets.

Definition 2.2.13. Let $f : Y \to X$ be a continuous function between two topological spaces, and let F be a sheaf of abelian groups defined on Y. Define the *direct image* of F along f to be the sheaf f_*F on X whose sections on an open set $U \subset X$ are

$$f_*F(U) = F(f^{-1}(U))$$

Remark 2.2.14. As is so often the case, I've hidden a proposition in the statement of this definition. It's easy to see that this formula defines a presheaf on X; you still need to work a little bit to verify that this presheaf satisfies the sheaf axioms.

Now we can correctly interpret the construction we started describing earlier. Given a ring homomorphism $\varphi : R \to S$, it not only determines a continuous function on spectra $f : Y = Spec(S) \to X = Spec(R)$, but it also defines a morphism of sheaves

$$f^{\#} : \mathcal{O}_X \to f_*\mathcal{O}_Y$$

by composing any section of the structure sheaf of X with f on one side and with the localizations of φ on the other.

Lemma 2.2.15: *With this notation, $f^{\#}_X : \mathcal{O}_X(X) \to \mathcal{O}_Y(Y = f^{-1}(X))$ is nothing other than the original ring homomorphism φ.*

Proof: By our earlier computation, we can at least see that $f^{\#}_X$ is a homomorphism from R to S. Now an element $r \in R$ is the section that assigns to each prime $P \subset R$ the element $r \in R_P$. Given any prime $Q \subset S$, the composition defining $f^{\#}(r)$ assigns to Q the element $\varphi(r) \in S_Q$. But this is exactly the same as the section corresponding to $\varphi(r) \in S$. ∎

Section 2.3 General Schemes

Definition 2.3.1. A *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ defined on X. A *morphism of ringed spaces* from $(Y, \mathcal{O}_Y)$ to $(X, \mathcal{O}_X)$ is a pair $(f, f^{\#})$ where $f : Y \to X$ is a continuous function and $f^{\#} : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is a morphism of sheaves on X.

Remark 2.3.2. We will often be careless about notation and talk about a ringed space X, tacitly assuming the existence of a structure sheaf $\mathcal{O}_X$. Similarly, we'll usually just write $f : Y \to X$ for a morphism of ringed spaces, assuming that $f^{\#}$ is the name of the corresponding morphism of sheaves.

Remark 2.3.3. We have already seen that the spectrum of a commutative ring is a ringed space; that's the motivation for this definition. In addition, every homomorphism of commutative rings induces a morphism of ringed spaces in the other direction. It is important to note, however, that these morphisms have an extra property. To describe it, let $\varphi : R \to S$ be a homomorphism of rings, inducing a morphism of ringed spaces $f : Y = Spec(S) \to X = Spec(R)$. Let $y \in Y$ be a point corresponding to a prime ideal $Q \subset S$. Write $x = f(y) \in X$ for the image, which corresponds to the prime ideal $P = \varphi^{-1}(Q) \subset R$. Look at an open neighborhood $x \in U \subset X$. Its inverse image $V = f^{-1}(U)$ is an open neighborhood of y in Y. Composing the sheaf maps with the map that computes germs at y gives a system of homomorphisms

$$germ_y \circ f^{\#}_U : \mathcal{O}_X(U) \to f_*\mathcal{O}_Y(U) = \mathcal{O}_Y(V) \to \mathcal{O}_{Y,y}.$$

Because these homomorphisms are compatible with the restriction homomorphisms, the univeral property of the direct limit produces a homomorphism

$$\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}.$$

The construction of these homomorphisms makes sense for arbitrary morphisms for ringed spaces. In the case of a morphism between ring spectra arising from an underlying homomorphism of rings, however, we can identify the stalks and the map more completely; it's just the ring homomorphism

$$\varphi_Q : R_P \to S_Q,$$

where $P = \varphi^{-1}(Q)$. The special property that this homomorphism of local rings has is that the inverse image of the maximal ideal QS_Q is the entire maximal ideal PR_P, and not some smaller prime ideal. A homomorphism of local rings with this property is called a *local homomorphism* . Not every ring homomorphism between local rings is a local homomorphism; consider, for example, the inclusion map $\mathbf{Z}_{(p)} \to \mathbf{Q}$ from the localization of the integers at the prime ideal (p) to the rational numbers.

Definition 2.3.4. A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ with the property that all of the stalks $\mathcal{O}_{X,x}$ are local rings. A *morphism of locally ringed spaces* is a morphism of ringed spaces

$$(f, f^{\#}) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

such that all the induced maps $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ are local homomorphisms. An *isomorphism* of locally ringed spaces is a morphism with a two-sided inverse.

Remark 2.3.5. Let X be a locally ringed space and let $U \subset X$ be an open subset. The functor $\mathcal{O}_X$ on the category $\mathcal{TOP}(X)^{op}$ restricts in a natural way to define a ring-valued functor $\mathcal{O}_U = \mathcal{O}_X|_U$ on $\mathcal{TOP}(U)^{op}$. The stalks of the restriction at points of U must be the same for these two functors, since they only depend on sufficiently small open neighborhoods of the point. Thus, $(U, \mathcal{O}_U)$ is also a locally ringed space.

Definition 2.3.6. A *scheme* is a locally ringed space X that has an open cover $\{U_i\}$ such that each U_i is isomorphic as a locally ringed space to $Spec(R_i)$ for some commutative ring R_i. A *morphism of schemes* is the same as a morphism of locally ringed spaces.

Definition 2.3.7. A scheme X is called an affine scheme if it is isomorphic (as a scheme, and hence as a locally ringed space) to $Spec(R)$ for some commutative ring R.

Example 2.3.8. Here is the simplest (and smallest) example of a non-affine scheme. Start with a discrete valuation ring R, and let K denote its field of fractions. The affine scheme $X = Spec(R)$ contains two points: The maximal ideal $M \subset R$ corresponds to a closed point $x \in X$; the zero ideal in R corresponds to the generic point $\eta \in X$, which is both open and dense. So, X contains exactly three open sets, and the structure sheaf is defined by

$$\begin{array}{ccc} \emptyset & \mapsto & 0 \\ \{\eta\} & \mapsto & K \\ X & \mapsto & R, \end{array}$$

with the obvious restriction maps. From this information, we can easily read off the stalks:

$$\mathcal{O}_{X,x} = R, \qquad \mathcal{O}_{X,\eta} = K.$$

Now start with two copies of X, and glue them together by identifying the two generic points. The result is a topological space Y containing three points, five open sets, and a structure sheaf

$$\begin{array}{rcl} \emptyset & \mapsto & 0 \\ \{\eta\} & \mapsto & K \\ \{\eta, x_1\} & \mapsto & R \\ \{\eta, x_2\} & \mapsto & R \\ Y = \{\eta, x_1, x_2\} & \mapsto & R. \end{array}$$

(The restriction maps are the obvious ones.) It's clear that this defines a scheme, since the two open subsets with two points are both scheme-theoretically isomorphic to $Spec(R)$. However, it can't be an affine scheme. After all, if Y were isomorphic to the affine scheme $Spec(S)$, then we would have $R = \mathcal{O}_Y(Y) = \Gamma(Y, \mathcal{O}_Y) = S$, which is impossible.

Let's now look at a more interesting class of examples. Start with a graded ring S. This means, among other things, that we can decompose $S = \oplus_{d \geq 0} S_d$ as a direct sum of abelian groups S_d. Elements of S_d are called *homogeneous elements* of degree d in S. Multiplication in S also satisfies: for each d, e we have $S_d.S_e \subset S_{d+e}$. We will write $S_+ = \oplus_{d>0} S_d$ for the ideal generated by all homogeneous elements of positive degree.

As a set, define $Proj(S)$ to be the set of homogeneous prime ideals $P \subset S$ that do not contain the ideal S_+. Given any homogeneous ideal $J \subset S$, we define the set $Z(J) = \{P \in Proj(S) : P \supset J\}$. As before, the $Z(J)$ are the closed sets of a topology (called the Zariski topology) on $Proj(S)$.

If P is a homogeneous prime ideal in S, write T for the multiplicative set of all **homogeneous** elements in $S \backslash P$. Because we are only inverting homogeneous elements, the localized ring $T^{-1}S$ has a natural grading. Write $S_{(P)}$ for the set of elements of degree zero in $T^{-1}S$. Now we can define a structure sheaf on $X = Proj(S)$ by taking the sections on an open subset U to be the set of functions $s : U \to \coprod_{P \in U} S_{(P)}$ such that for each $P \in U$, one has $s(P) \in S_{(P)}$ and there exists an open neighborhood V of P in U and homogeneous elements a, f in S of the same degree such that for all $Q \in V$, one has $s(Q) = a/f \in S_{(Q)}$.

Phew. Of course, that's exactly the sort of definition we've seen twice before. It's clear that it gives a sheaf of rings, making $Proj(S)$ into a ringed space.

Theorem 2.3.9: *With these definitions, $X = Proj(S)$ is a scheme.*

Proof: One needs to verify these facts:

(i) The stalk $\mathcal{O}_{X,P}$ is naturally isomorphic to the degree zero localization $S_{(P)}$.

(ii) For any homogeneous element $f \in S_+$, let $D_+(f) = Proj(S) \backslash Z(f)$. Each $D_+(f)$ is open in $Proj(S)$, and the collection of all $D_+(f)$ covers $Proj(S)$.

(iii) For any homogeneous element $f \in S_+$, let $S_{(f)}$ be the subring of elements of degree zero in the localized ring S_f. There is an isomorphism of locally ringed spaces between $D_+(f)$ and $Spec(S_{(f)})$.

(i) Any element of the stalk is represented in some neighborhood as a degree zero fraction, and thus gives rise to an element of the localized ring. This construction clearly defines a surjection. The proof of injectivity is then similar to the argument we gave in the affine case. Note that, as a consequence, we can conclude that $Proj(S)$ is a locally ringed space.

(ii) The $D_+(f)$ are clearly open. If P is a homogeneous prime ideal that does not contain all of S_+, then we can choose an element $f \in S_+ \backslash P$. Then $P \in D_+(f)$.

(iii) Given any homogeneous ideal J, define $\varphi(J) = (JS_f) \cap S_{(f)}$. When P is prime and $f \notin P$, then $\varphi(P)$ is also prime, thus defining a set-theoretic map $D_+(f) \to Spec(S_{(f)})$. The properties of localization show that this is a bijection; by looking at properties of containment, one also sees that it is a homeomorphism. Finally, the stalks of the structure sheaves on these spaces are given by the isomorphic local rings $S_{(P)}$ and $(S_{(f)})_{\varphi(P)}$. ∎

Example 2.3.10. Let R be an arbitrary commutative ring, and let $S = R[X_0, X_1, \ldots, X_n]$ be the polynomial ring over R with its usual grading. Define *projective n-space* over R to be $\mathbf{P}^n_R = Proj(S)$. In particular, if k is an algebraically closed field, one can check that the closed points of this scheme are exactly the same as the points of the variety we previously constructed. In the general case, we have a finite open cover given by the affine open subsets $D_+(X_i) \sim R[X_0, \ldots, X_n]_{(X_i)}$. Fixing the index i, we can write $x_j = X_j/X_i$; it is easy to see that the degree zero localization is isomorphic to $R[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$. Thus, $D_+(X_i)$ is isomorphic to affine n-space, $\mathbf{A}^n_R$.

Proposition 2.3.11: *Let X be an arbitrary scheme and let R be a commutative ring. Then there is a natural bijection:*

$$\alpha : Mor(X, Spec(R)) \to Hom(R, \Gamma(X, O_X)).$$

Proof: Given a morphism $f : X \to Y = Spec(R)$, the sheaf-theoretic part of the morphism induces a map on global section by

$$\alpha(f) : R = \mathcal{O}_Y(Y) \to f_*\mathcal{O}_X(Y) = \mathcal{O}_X(f^{-1}(Y)) = \mathcal{O}_X(X) = \Gamma(X, \mathcal{O}_X).$$

Thus, to any scheme morphism we can associate a ring homomorphism. To go the other direction, start with a ring homomorphism $\varphi : R \to \Gamma(X, \mathcal{O}_X)$. Let $U = Spec(A) \subset X$ be any affine open subset. Composing with the restriction map ρ^X_U, we get a ring homomorphism $R \to A$. As before, this allows us to construct a morphism of schemes $Spec(A) \to Spec(R)$. It is clear that if $V = Spec(B)$ is another affine open subset contatined in U, then the chain of ring homomorphisms $R \to A \to B$ induces compatible scheme morphisms $Spec(B) \to Spec(A) \to Spec(R)$. Thus, we have a morphism defined on each open affine of X, and these morphisms are compatible on affine subsets inside the intersection of two affines. We can glue these together to produce a well-defined morphism on all of X. It is now straightforward to check that the two constructions are inverses, yielding the desired bijection. ∎

Remark 2.3.12. I've cheated here, because I've hidden one of the key points: this is where the local homomorphism property of maps between locally ringed spaces needs to be used.

Corollary 2.3.13: *Every morphism of affine schemes is induced from a ring homomorphism.*

Corollary 2.3.14: *The functor $R \mapsto Spec(R)$ is a fully faithful contravariant functor from the category RG of commutative rings into the category SCH of schemes; the image is naturally equivalent to the category of affine schemes.*

Proof: We already know that we can recover the ring R (up to isomorphism) from the scheme $Spec(R)$ by looking at the global sections of its structure sheaf. The previous proposition says that every morphism between affine schemes arises in a unique way from a ring homomorphism, yielding the result. ∎

Corollary 2.3.15: *$Spec(\mathbf{Z})$ is a final object in the category of schemes.*

Proof: Recall that an object in a category is called *final* if there is a unique morphism to that object from every other object in the category. Well,

$$Mor(X, Spec(\mathbf{Z})) = Hom(\mathbf{Z}, \Gamma(X, O_X)),$$

and there is a unique ring homomorphism from $\mathbf{Z}$ to any other commutative ring with unity. ∎

Section 2.4 Pointed Remarks

In this section, we want to sort out the various meanings of the word "point," as used by algebraic geometers. This word is used in at least three distinct ways. As a result, beginners in the subject often find themselves confused.

Let k be a field, and let S be an algebraic variety over k. (An *algebraic variety* is a scheme of finite type over a field. This usage differs from that of other authors, who reserve the word variety for an irreducible scheme of finite type. We want to work over arbitrary fields, and the property of being irreducible is not stable under the operation of extending the base field.) The most primitive notion of a point is also the rarest usage among algebraic geometers. We call any element of the underlying topological space of S a point. When using the term this way, we often make special mention of the *closed points* . This term emphasizes the fact that the underlying topological space is not Hausdorff, and many of its points are, in fact, not closed in S.

Let's assume for the moment that the variety in question can be embedded in a projective space $\mathbf{P}^n_k$. Then we can give an elementary definition of a k-rational point of S. A closed point $s \in S \subset \mathbf{P}^n$ is called a *k-rational point* if it has the form $s = (s_0 : \cdots : s_n)$ with all $s_i \in k$. We write $S(k)$ for the set of all k-rational points of the scheme S.

If the field k is not algebraically closed, then the k-rational points usually form a very small subset of the collection of all closed points of S. We can reinterpret the definition of a k-rational point in a coordinate-free manner that will allow us to generalize the above definition in several ways. Notice that to give a k-rational point of S (in the sense described above) is equivalent to specifying a k-morphism $Spec(k) \to S$. The first generalization, then, is to drop the assumption that S can be embedded in projective space. Next, suppose that $s \in S$ is a closed point. Then the residue field $k(s) = \mathcal{O}_{S,s}/m_s$ is an extension field of k of finite degree. The inclusion of this closed point in S corresponds (loosely) to a k-morphism $Spec(k(s)) \to S$. The looseness of the correspondence comes from the fact that we can compose this morphism with any k-automorphism of $k(s)$ without detecting any difference in the closed point $s \in S$. To elaborate, let L/k be any field extension of finite degree. Let us define an L-rational point of the k-scheme S to be a k-morphism $Spec(L) \to S$, and denote the set of all L-rational points of S by $S(L)$. In general, the set $S(L)$ is strictly larger than the set of closed points of S whose coefficients lie in L.

Example 2.4.1. Let $k = \mathbf{R}$ be the field of real numbers, let $L = \mathbf{C}$ be the field of complex numbers, and let $S = Spec(\mathbf{R})$. Now S is an $\mathbf{R}$-scheme with one closed point, and $S(\mathbf{R})$ is a one-point set. After all, there is only one $\mathbf{R}$-algebra map from the reals into itself. Moreover, the set $S(\mathbf{C})$ also consists of exactly one point, because any $\mathbf{R}$-algebra homomorphism from $\mathbf{R}$ to $\mathbf{C}$ must take the number 1 to itself.

Example 2.4.2. Now consider $S = Spec(\mathbf{C})$ as a scheme over the field $\mathbf{R}$. This is another scheme that has only one closed point. However, $S(\mathbf{R})$ is the empty set (since there are no homomorphisms $\mathbf{C} \to \mathbf{R}$) and $S(\mathbf{C})$ is a two-point set (since there are two $\mathbf{R}$-algebra maps from $\mathbf{C}$ to itself, depending on whether $i \mapsto i$ or $i \mapsto -i$).

Example 2.4.3. Now take $S = Spec(\mathbf{R}[t]) = \mathbf{A}^1_{\mathbf{R}}$ as a scheme over $\mathbf{R}$. The set of closed points of S can be naturally identified with the complex upper half plane, the set $S(\mathbf{R})$ with the real axis, and the set $S(\mathbf{C})$ with the entire complex plane.

We are now ready to define yet another kind of point on a k-scheme S. There is no reason to restrict the previous definition to finite extension fields of k. If we let $\bar{k}$ denote the algebraic closure of k, then we can consider the set

$$S(\bar{k}) = \{\alpha : Spec(\bar{k}) \to S\} = Hom_k(Spec(\bar{k}), S).$$

An element of this set is often called a *geometric point* of S. Quite often, an algebraic geometer will use the word "point" to refer to any geometric point of a variety.

Remark 2.4.4. One nice aspect of the definition of rational points with values in extension fields is that it behaves well under base extension. If S is a k-scheme and if L/k is any extension field, we write $S_L = Spec(L) \times_{Spec(k)} S$ for the L-scheme obtained by base extension. It is easy to use the universal properties of fibre products to show that there is a natural bijection $S_L(L) = S(L)$. (You should convince yourself that this is true in the examples given above.)

Now we come to the most general notion of a point. Let S and T be arbitrary schemes over a field k. The set of T-valued points of S is defined to be

$$S(T) = Hom_k(T, S).$$

This new notion does not look like much of a leap beyond our earlier definitions, but it has some fascinating consequences. We have gotten here by considering more and more general notions of the idea of a point. We have ended up considering points with values in an arbitrary k-scheme. Moreover, suppose that we have a morphism $f : T \to U$ of k-schemes. Composition of morphisms defines a function

$$\circ f : S(U) = Hom_k(U, S) \to Hom_k(T, S) = S(T).$$

This gives us a completely new way to look at a scheme: A scheme S gives rise to a functor, which assigns to any k-scheme T a certain set. We call this new thing the *functor of points* associated to the scheme S.

Example 2.4.5. Take $S = Spec(k)$. Since

$$S(T) = Mor_k(T, S) = Hom_k(k, \Gamma(T, \mathcal{O}_T))$$

contains exactly one element for each k-scheme T, the scheme S corresponds to the constant functor $T \mapsto \{*\}$.

Example 2.4.6. Take $S = \mathbf{A}^1_k = Spec(k[x])$. What functor is defined by the affine line? Since this is an affine scheme, we have

$$S(T) = Mor_k(T, S) = Hom_k(k[x], \Gamma(T, \mathcal{O}_T)) = \Gamma(T, \mathcal{O}_T).$$

Thus, the affine line over k is equivalent to the functor $T \mapsto \Gamma(T, \mathcal{O}_T)$. In this case, the functor takes values not just in the category of sets, but in the category of k-algebras; this observation suggests that there is more structure to the affine line than we have so far used.

Example 2.4.7. Take $S = Spec(k[x, x^{-1}])$. This is another affine scheme, so we have

$$S(T) = Mor_k(T, S) = Hom_k(k[x, x^{-1}], \Gamma(T, \mathcal{O}_T)) = \Gamma(T, \mathcal{O}_T)^*.$$

Thus, S is equivalent to the functor that maps a scheme T to the multiplicative group $\Gamma(T, \mathcal{O}_T)^*$. For this reason, this scheme is often denoted $\mathbf{G}_m$.

Let us denote the functor of points of S by the notation h_S, so that $h_S(T) = S(T) = Hom_k(T, S)$. In this way, we can distinguish the scheme S from the functor h_S. Now we realize that there is another functor staring us in the face. The assignment $S \mapsto h_S$ defines a functor

$$h : \mathcal{SCH}_k \to \mathcal{FUN}(\mathcal{SCH}_k^{op}, \mathcal{SET}).$$

Proposition 2.4.8: *The functor h is fully faithful.*

Proof: Let's recall what this statement means. For the functor h to be full, we need to show that for any two schemes S and T, any natural transformation $\eta : h_S \to h_T$ comes from a map of schemes $S \to T$. But this is obvious: take the identity map of S as an element of $h_S(S)$ and apply η to get an element $\eta_S(1_S) \in h_T(S) = Hom(S,T)$. It is easy to use the naturality of η to show that it must be given by composing with the morphism $\eta_S(1_S) : S \to T$.

For the functor h to be faithful, we must show that if two morphisms of schemes $f, g : S \to T$ induce the same natural transformation $h_S \to h_T$, then they must have started out as the same morphism. But this is also easy, since the natural transformation is defined by composing with the morphism, and we can test it on an identity morphism. ∎

After continually generalizing the notion of points, we are going to restrict it slightly. Suppose we restrict our attention to affine k-schemes $T = Spec(R)$, and just look at R-valued points of an arbitrary k-scheme S. Then we have a functor

$$h^0 : \mathcal{SCH}_k \to \mathcal{FUN}(\mathcal{ALG}_k, \mathcal{SET}),$$

defined by $S \mapsto h^0_S$, where $h^0_S(R) = h_S(Spec(R))$.

Proposition 2.4.9: *The functor h^0 is fully faithful.*

Proof: Left to the careful and industrious reader. One should note that you need to glue morphisms together on affine patches to show that h^0 is full. ∎

Remark 2.4.10. The fully faithful functors h and h^0 allow us to identify the category of schemes with a subcategory of a functor category. That may sound like abstract nonsense, but it has practical implications. In particular, we have shown that the functor of points determines the scheme. In other words, if we know enough about the points on a scheme, then we know everything about the scheme in question.